Trinity College

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

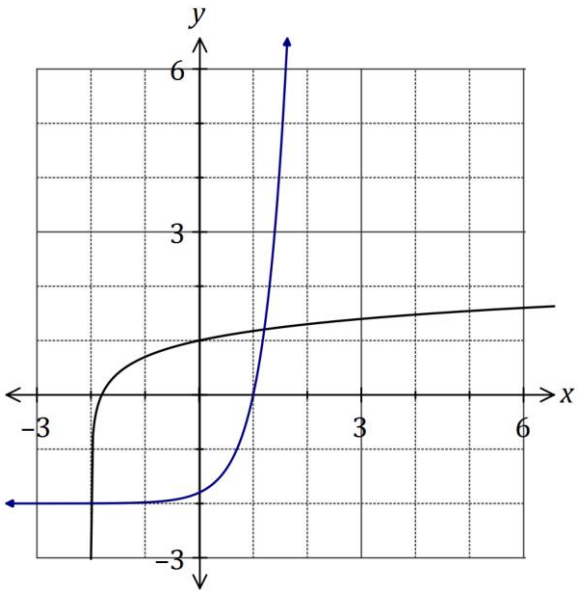
Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = \log(5x + 10)$.



(a) Explain why y is a function of x over the natural domain of f. (1 mark)

Solution
A one-to-one relationship clearly exists.
Specific behaviours
✓ reasonable explanation

(b) Determine a function for the inverse of f. (2 marks)

Solution
Let $x = \log(5y + 10)$ $10^x = 5y + 10$ $y = f^{-1}(x) = \frac{10^x - 10}{5}$
Specific behaviours
✓ eliminates log ✓ expresses as function of x

(c) On the axes above, sketch the graph of the inverse of f. (2 marks)

Solution
See graph
Specific behaviours
✓ x and y intercepts ✓ clearly reflection of f in line $y = x$

See next page

Question 2 **(7 marks)**

(a) Given that $z - 1 + i$ is a factor of $z^4 + az^2 + bz$, determine the values of the real constants a and b. (4 marks)

Solution
Let $z = 1 - i$ $z^2 = (1 - i)(1 - i) = -2i$ $z^4 = (-2i)^2 = -4$ $-4 - 2ai + b - bi = 0$ Re: $-4 + b = 0 \Rightarrow b = 4$ Im: $-2a - 4 = 0 \Rightarrow a = -2$ $a = -2, \quad b = 4$
Specific behaviours
✓ identifies root and substitutes ✓ determines z^4 correctly ✓ equates Re and Im parts ✓ solves for constants

(b) Solve the equation $z^4 - z^2 - 20 = 0$. (3 marks)

Solution
$(z^2 - 5)(z^2 + 4) = 0$ $z = \pm\sqrt{5}, \quad z = \pm 2i$
Specific behaviours
✓ factorises ✓ Re solutions ✓ Im solutions

See next page

Question 3 **(7 marks)**

Let $u = \sqrt{3} - i$ and $z = 2 + 2i$.

(a) Determine the argument of $z \div u^3$. (3 marks)

Solution
$\arg u^3 = 3 \times -\frac{\pi}{6} = -\frac{\pi}{2}$ $\arg z = \frac{\pi}{4}$ $\arg(z \div u^3) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$
Specific behaviours
✓ indicates $\arg u^3$ and $\arg z$ ✓ indicates subtraction of arguments ✓ states required argument

(b) Determine the real part of $(iz)^4$. (4 marks)

Solution
$iz = -2 + 2i$ $= 2\sqrt{2}\,\text{cis}\frac{3\pi}{4}$ $(iz)^4 = 2^6\,\text{cis}\,\pi$ $\text{Re part} = -2^6 = -64$
Specific behaviours
✓ indicates iz ✓ expresses iz in polar form ✓ determines $(iz)^4$ ✓ states real part

See next page

Question 4 **(7 marks)**

(a) Determine the slope of the graph of $2x^2 + y^2 = 3xy$ at the point $(1, 2)$. (3 marks)

Solution
$4x + 2yy' = 3y + 3xy'$ $4 + 4y' = 6 + 3y'$ $y' = 2$
Specific behaviours
✓ differentiates LHS ✓ differentiates RHS ✓ substitutes and simplifies

(b) Use the substitution $u = 5x^2 - 4$ to express $\int_1^2 \frac{10x}{\sqrt{5x^2 - 4}}\, dx$ in terms of u and hence evaluate the integral. (4 marks)

Solution
$u = 5x^2 - 4 \Rightarrow du = 10x\, dx$ $x = 1, u = 1;\ \ x = 2, u = 16$ $\int_1^2 \frac{10x}{\sqrt{5x^2 - 4}}\, dx = \int_1^{16} \frac{1}{\sqrt{u}}\, du$ $= \left[2\sqrt{u}\right]_1^{16} = 8 - 2 = 6$
Specific behaviours
✓ expresses integrand in terms of u ✓ changes limits ✓ antidifferentiates ✓ evaluates

Question 5 **(6 marks)**

Two functions are defined by

$$f(x) = \frac{x-2}{x^2-4} \quad \text{and} \quad g(x) = x + 7.$$

(a) Determine $g \circ f(1)$. (1 mark)

Solution
$f(1) = \frac{1}{3}, \quad g\left(\frac{1}{3}\right) = \frac{22}{3}, \quad g \circ f(1) = \frac{22}{3}$
Specific behaviours
✓ evaluates

(b) Determine a simplified expression for $f \circ g(x)$ and state the domain and range for this composite function. (5 marks)

Solution
$f(x) = \frac{x-2}{(x+2)(x-2)}$ $f \circ g(x) = \frac{x+7-2}{(x+7+2)(x+7-2)}$ $= \frac{1}{x+9}, x \neq -5$ $D_{fog} = \{x: x \neq -9, x \neq -5\}$ $R_{fog} = \left\{y: y \neq 0, y \neq \frac{1}{4}\right\}$
Specific behaviours
✓ correct expression for composite ✓ simplifies composite ✓ correct domain ✓ range: includes $y \neq 0$ ✓ correct range

See next page

Question 6 **(5 marks)**

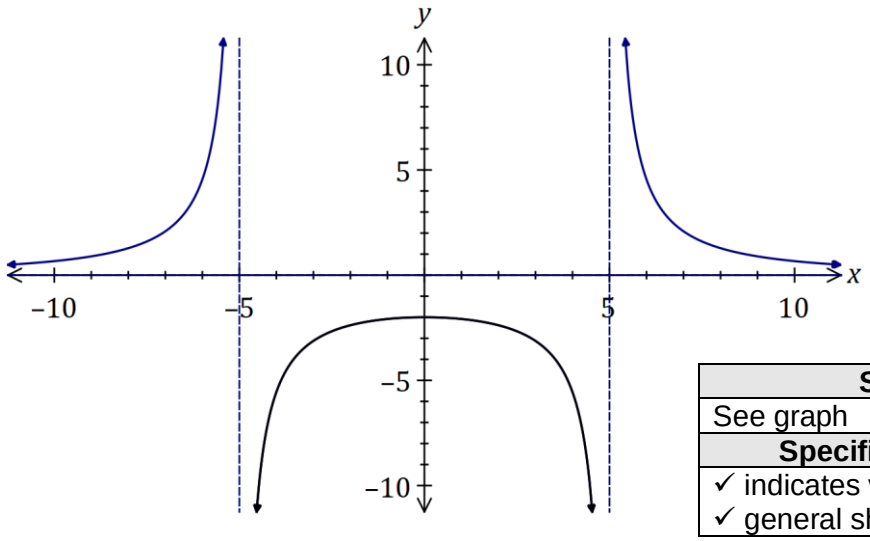
A plane passes through the points $A(0, 0, 1)$, $B(1, -2, 3)$ and $C(3, 1, -1)$. Determine the Cartesian equation of the plane.

Solution
$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix} = 7$ $2x + 8y + 7z = 7$
Specific behaviours
✓ determines two vectors in plane ✓ indicates use of cross product to find normal ✓ correct normal ✓ determines constant d ✓ states equation in Cartesian form

Question 7 **(7 marks)**

(a) Part of the graph with equation $y = \dfrac{50}{x^2 - 25}$ is shown below.

Complete the sketch, clearly indicating the location of any asymptotes. (2 marks)



Solution
See graph
Specific behaviours
✓ indicates vertical asymptotes ✓ general shape

(b) Determine the area bounded by $y = \dfrac{50}{x^2 - 25}$, the x-axis, the y-axis and the line $x = 4$.

(5 marks)

Solution

$$\frac{50}{x^2 - 25} = \frac{50}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$$

$$A(x+5) + B(x-5) = 50 \Rightarrow A = 5, B = -5$$

$$\int_0^4 \frac{5}{x-5} - \frac{5}{x+5}\,dx = [5\ln|x-5| - 5\ln|x+5|]_0^4$$
$$= [0 - 5\ln 9] - [5\ln 5 - 5\ln 5]$$

$$A = 5\ln 9 \ (= 10\ln 3)$$

Specific behaviours

✓ indicates use of partial fractions
✓ correct partial fractions
✓ indicates definite integral
✓ integrates correctly, using absolute values
✓ correct area

See next page

Question 8 **(8 marks)**

(a) Determine all solutions to the equation $2^7 z^6 = -\sqrt{2} + \sqrt{2}i$ in the form $r\,\text{cis}\,\theta$ where $r > 0$ and $-\pi < \theta \leq \pi$. (5 marks)

Solution
$2^7 z^6 = 2\,\text{cis}\dfrac{3\pi}{4}$ $z^6 = 2^{-6}\,\text{cis}\dfrac{3\pi}{4}$ $\text{z} = \dfrac{1}{2}\text{cis}\left(\dfrac{\pi}{8} + \dfrac{k\pi}{3}\right), k = -3, -2, -1, 0, 1, 2$ $z_1 = \dfrac{1}{2}\text{cis}\dfrac{-7\pi}{8}$ $z_2 = \dfrac{1}{2}\text{cis}\dfrac{-13\pi}{24}$ $z_3 = \dfrac{1}{2}\text{cis}\dfrac{-5\pi}{24}$ $z_4 = \dfrac{1}{2}\text{cis}\dfrac{\pi}{8}$ $z_5 = \dfrac{1}{2}\text{cis}\dfrac{11\pi}{24}$ $z_6 = \dfrac{1}{2}\text{cis}\dfrac{19\pi}{24}$
Specific behaviours
✓ expresses z^6 in polar form ✓ uses De Moivre's theorem ✓ expresses general solution in terms of k ✓ states one correct root ✓ states all six roots correctly

(b) If w is any complex cube root of unity, simplify $(1 + 4w)(1 + 4w^2)$. (3 marks)

Solution
$(1 + 4w)(1 + 4w^2) = 1 + 4w + 4w^2 + 16w^3$ But $w^3 = 1$ and $1 + w + w^2 = 0$ hence $(1 + 4w)(1 + 4w^2) = -3 + 4(1 + w + w^2) + 16(1) = 13$
Specific behaviours
✓ uses cube of root ✓ uses sum of roots ✓ simplifies

End of questions

Additional working space

Question number: _____